

Calculus III: Coursework 8
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The deadline is 5PM on Tuesday December 5th. Hand-in is to the green box in the basement of the Maths building.

Notes to students:

1. This is the last *required* coursework.
2. Questions 1 and 2 are adapted from old exam questions. The originals asked for statements of the theorems concerned - note that that would be a reasonable thing to ask in an exam even though proofs are not required.

1. Evaluate both sides of the Divergence Theorem for the field $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + 0\mathbf{k}$ over the box bounded by the planes $x = 0, a, y = 0, b, z = 0, c$ and hence show that the two sides are equal. [22]

Answer

$\nabla \cdot \mathbf{F} = y + z$ so $\int \nabla \cdot \mathbf{F} \, dV$ over the required region is

$$\begin{aligned} \int_0^c \int_0^b \int_0^a (y + z) \, dx \, dy \, dz &= \int_0^c \int_0^b a(y + z) \, dy \, dz \\ &= \int_0^c (a[\frac{1}{2}y^2]_0^b + az[y]_0^b) \, dz \\ &= \int_0^c (\frac{1}{2}ab^2 + abz) \, dz \\ &= \frac{1}{2}ab^2[z]_0^c + \frac{1}{2}ab[z^2]_0^c = \frac{1}{2}abc(b + c) \end{aligned}$$

The surface integral has 6 parts, one for each face. On $x = a$, $\mathbf{n} = \mathbf{i}$ and $\mathbf{F} \cdot \mathbf{n} = xy = ay$ so this face contributes

$$\int_0^c \int_0^b ay \, dy \, dz = \int_0^c \frac{1}{2}ab^2 \, dz = \frac{1}{2}ab^2c.$$

On $x = 0$, $\mathbf{F} \cdot \mathbf{n} = 0$, so there is no contribution, and similarly on $y = 0, z = 0, c$. We still have $y = b$ to do, where $\mathbf{n} = \mathbf{j}$ and $\mathbf{F} \cdot \mathbf{n} = bz$. This contributes

$$\int_0^c \int_0^a bz \, dx \, dz = \frac{1}{2}abc^2.$$

Adding the two non-zero contributions we get the same result as before.

Marks: 3 for the divergence, 6 for volume integration, 3 for knowing how to get correct normals etc on at least one face, 1 each for the four faces giving zero, 3 each for the other two faces.

2. Evaluate both sides of Stokes's Theorem for the same vector field as in the previous question, for a closed path in the $z = 0$ plane consisting of straight path segments as follows: from the origin to $(a, 0, 0)$; from there to $(a, b, 0)$; from there to $(0, b, 0)$; and back to the origin. Hence show that the two sides are equal.

Answer

$\nabla \times \mathbf{F} = -y\mathbf{i} - x\mathbf{k}$ and the surface integral is over a rectangle in the (x, y) plane with $\mathbf{n} = \mathbf{k}$. Its value is

$$\int_0^b \int_0^a (-x) dx dy = -\frac{1}{2}a^2b.$$

[18]

The integral along the curve has four parts.

1. $\mathbf{r} = t\mathbf{i}$, $0 \leq t \leq a$. Here $\mathbf{F} \cdot d\mathbf{r} = 0$.

2. $\mathbf{r} = a\mathbf{i} + t\mathbf{j}$, $0 \leq t \leq b$. Here again $\mathbf{F} \cdot d\mathbf{r} = 0$.

3. $\mathbf{r} = (a-t)\mathbf{i} + b\mathbf{j}$, $0 \leq t \leq a$. Here $d\mathbf{r} = -\mathbf{i}$ and $\mathbf{F} \cdot d\mathbf{r} = -xy = -(a-t)b$. This contributes

$$\int_0^a (t-a)b dt = \frac{1}{2}a^2b - a^2b = -\frac{1}{2}a^2b.$$

4. $\mathbf{r} = (b-t)\mathbf{j}$, $0 \leq t \leq b$. Yet again $\mathbf{F} \cdot d\mathbf{r} = 0$.

So the total agrees with the first evaluation.

Marks: 3 for the curl, 6 for the surface integration, 3 for correct parametrization of at least one part of the curve, 1 each for the zero contributions, 3 for the non-zero one.

3. In spherical polar coordinates (r, θ, ϕ) , evaluate

(a) ∇f

(b) $\nabla \cdot \mathbf{F}$

(c) $\nabla \times \mathbf{F}$

where $f = r^2 \sin \theta \cos \phi$ and $\mathbf{F} = r \sin \theta \mathbf{e}_\phi$.

[12]

Answer

[This and the next question just require substitution into the formulae collected at the end of section 7.7 in the notes. Part (a) of this first one is very like Examples 6.5/6.6 in the notes, and part (c) follows from Example 7.11.]

(a)

$$\begin{aligned} \nabla f &= \frac{\partial(r^2 \sin \theta \cos \phi)}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial(r^2 \sin \theta \cos \phi)}{\partial \theta} \mathbf{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial(r^2 \sin \theta \cos \phi)}{\partial \phi} \mathbf{e}_\phi \\ &= 2r \sin \theta \cos \phi \mathbf{e}_r + r \cos \theta \cos \phi \mathbf{e}_\theta - r \sin \phi \mathbf{e}_\phi. \end{aligned}$$

(b)

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2 \sin \theta} \left[\frac{\partial(r^2 \sin \theta F_r)}{\partial r} + \frac{\partial(r \sin \theta F_\theta)}{\partial \theta} + \frac{\partial(r F_\phi)}{\partial \phi} \right] = \frac{1}{r^2 \sin \theta} \frac{\partial(r^2 \sin \theta)}{\partial \phi} = 0.$$

(c) Using the result of the example mentioned above,

$$\nabla \times \mathbf{F} = \frac{1}{r^2 \sin \theta} \left[\frac{\partial(r^2 \sin^2 \theta)}{\partial \theta} \right] \mathbf{e}_r + \frac{1}{r \sin \theta} \left[-\frac{\partial(r^2 \sin^2 \theta)}{\partial r} \right] \mathbf{e}_\theta = 2 \cos \theta \mathbf{e}_r - 2 \sin \theta \mathbf{e}_\theta.$$

Marks: 3 for the gradient, 4 for the divergence, 5 for the curl (no marks for any part where the answer should be a scalar and is given as a vector, or vice versa).

4. In cylindrical polar coordinates (ρ, ϕ, z) , evaluate

(a) ∇f

(b) $\nabla \cdot \mathbf{F}$

(c) $\nabla \times \mathbf{F}$

where $f = \rho^2 z \sin \phi$ and $\mathbf{F} = \left(\frac{1}{\rho} + \cos \phi\right) \mathbf{e}_\rho + \sin \phi \mathbf{e}_\phi$. [12]

Answer

[Example 7.10 in the notes helps with part (b) here.]

(a)

$$\nabla f = \frac{\partial f}{\partial \rho} \mathbf{e}_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \mathbf{e}_\phi + \frac{\partial f}{\partial z} \mathbf{e}_z = 2\rho z \sin \phi \mathbf{e}_\rho + \rho z \cos \phi \mathbf{e}_\phi + \rho^2 \sin \phi \mathbf{e}_z.$$

(b) Using the example,

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho} \left[\frac{\partial(1 + \rho \cos \phi)}{\partial \rho} + \frac{\partial \sin \phi}{\partial \phi} \right] = \frac{2 \cos \phi}{\rho}.$$

(c)

$$\nabla \times \mathbf{F} = \frac{1}{\rho} \begin{vmatrix} \mathbf{e}_\rho & \rho \mathbf{e}_\phi & \mathbf{e}_z \\ \partial/\partial \rho & \partial/\partial \phi & \partial/\partial z \\ F_\rho & \rho F_\phi & F_z \end{vmatrix} = \frac{1}{\rho} \begin{vmatrix} \mathbf{e}_\rho & \rho \mathbf{e}_\phi & \mathbf{e}_z \\ \partial/\partial \rho & \partial/\partial \phi & \partial/\partial z \\ \left(\frac{1}{\rho} + \cos \phi\right) & \rho \sin \phi & 0 \end{vmatrix} = \frac{2 \sin \phi}{\rho} \mathbf{e}_z.$$

Marks: 3 for the gradient, 4 for the divergence, 5 for the curl (no marks for any part where the answer should be a scalar and is given as a vector, or vice versa).

5. A vector field is given in spherical polar coordinates by

$$\mathbf{F} = r^2 \sin \theta \mathbf{e}_r + 2r^2 \cos \theta \mathbf{e}_\theta - r^2 \sin(2\theta) \mathbf{e}_\phi.$$

Show that $\nabla \cdot \mathbf{F} = 4r \sin^2 \theta / \sin \theta$.

Evaluate both the integrals in the Divergence Theorem for $\mathbf{F}$ and a sphere of radius a centred at the origin, and show they are equal.

A second vector field differs from $\mathbf{F}$ only in its component along $\mathbf{e}_\phi$ and is given by

$$\mathbf{F}_2 = r^2 \sin \theta \mathbf{e}_r + 2r^2 \cos \theta \mathbf{e}_\theta + 2r^2(\sin \theta - 1)\phi \mathbf{e}_\phi.$$

Show that this vector field **does not satisfy** the Divergence Theorem. Explain why, and comment on the validity of using the Divergence Theorem for $\mathbf{F}$ itself. [24]

Answer

$$\begin{aligned}
\nabla \cdot \mathbf{F} &= \frac{1}{r^2 \sin \theta} \left[\frac{\partial(r^2 \sin \theta F_r)}{\partial r} + \frac{\partial(r \sin \theta F_\theta)}{\partial \theta} + \frac{\partial(r F_\phi)}{\partial \phi} \right] \\
&= \frac{1}{r^2 \sin \theta} \left[\frac{\partial(r^4 \sin^2 \theta)}{\partial r} + \frac{\partial(2r^3 \sin \theta \cos \theta)}{\partial \theta} + \frac{\partial(-r^3 \sin(2\phi))}{\partial \phi} \right] \\
&= \frac{1}{r^2 \sin \theta} [4r^3 \sin^2 \theta + 2r^3(\cos^2 \theta - \sin^2 \theta) - 2r^3 \cos(2\phi)] \\
&= 2r(1 - \cos(2\phi))/\sin \theta = 4r \sin^2 \phi / \sin \theta.
\end{aligned}$$

Using the formula for the area element in spherical polars (section 6.3 in the notes) the surface integral is

$$\int_{r=a} \mathbf{F} \cdot (r^2 \sin \theta \, d\theta \, d\phi \, \mathbf{e}_r) = \int_0^{2\pi} \int_0^\pi a^4 \sin^2 \theta \, d\theta \, d\phi = \pi^2 a^4,$$

while the volume integral (using the volume element $r^2 \sin \theta \, dr \, d\theta \, d\phi$) is

$$\int_0^{2\pi} \int_0^\pi \int_0^a 4r^3 \sin^2 \phi \, dr \, d\theta \, d\phi = \pi^2 a^4.$$

$\mathbf{F}_2$ will give the same surface integral as $\mathbf{F}$ since only the $\mathbf{e}_r$ component contributes. The divergence will now be

$$\frac{1}{r^2 \sin \theta} [4r^3 \sin^2 \theta + 2r^3(\cos^2 \theta - \sin^2 \theta) + 2r^3(\sin \theta - 1)] = 2r$$

and the volume integral is

$$\int_0^{2\pi} \int_0^\pi \int_0^a 2r^3 \sin \theta \, dr \, d\theta \, d\phi = 2\pi a^4.$$

The problem arises because the second vector field has a jump discontinuity in its $\mathbf{e}_\phi$ component at $\phi = 0 \equiv \phi = 2\pi$ (unless $\sin \theta = 1$), so the conditions of the theorem are not fulfilled. The divergence of the first field is not well defined at $\sin \theta = 0$ (i.e. on the polar axis) so this also runs into trouble in principle: it just so happens that the contributions to the integral from the singular points are zero, i.e. the improper integrals are well defined.

Marks: 4 for the divergence, 4 for the volume integral, 4 for the surface integral, 3 for the new divergence, 4 again for the volume integral, 3+2 for correct remarks about the two cases.

(Marker: there may be other ways to see why the Theorem does not apply so please read answers with care.)

6. A cylinder is described in cylindrical coordinates (ρ, ϕ, z) by $\rho \leq 2, 0 \leq z \leq 3$. Let

$$\mathbf{F} = \rho z \sin \phi \cos \phi \mathbf{e}_\rho - \rho z \sin^2 \phi \mathbf{e}_\phi + z \rho \cos \phi \mathbf{e}_z.$$

Evaluate, using Stokes' theorem or otherwise, the integral $\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$ over the surface S consisting of the base ($z = 0$) and the curved side of the cylinder, with normal taken downwards on the base. [12]

Answer

By Stokes's theorem the surface integral is the same as the line integral $\oint \mathbf{F} \cdot d\mathbf{r}$ round the circle $\mathcal{C}$ given by $\rho = 2, z = 3$, and taken clockwise (to agree with the direction for the normal to the base). On the circle $d\rho = -\rho d\phi \mathbf{e}_\phi = -2d\phi \mathbf{e}_\phi$ so we need only the $\mathbf{e}_\phi$ component of $\mathbf{F}$ which is $-6 \sin^2 \phi$ there, so we have

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 12 \int_0^{2\pi} \sin^2 \phi d\phi = 12\pi.$$

Marks: Use of Stokes and correct curve 4, parametrization of curve (including correct sign) 4, completion of integral 4